TWMS J. Pure Appl. Math., V.5, N.1, 2014, pp.50-58

ON THE EXISTENCE AND UNIQUENESS OF SOLUTION FOR A CLASS OF FRACTIONAL ORDER BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we discuss existence and uniqueness of solutions to nonlinear fractional order ordinary differential equations with boundary conditions in an ordered Banach space. We use the Caputo fractional differential operator and the nonlinearity depends on the fractional derivative of an unknown function. Schauder's fixed point Theorem is the main tool used here to establish the existence and the Banach contraction principle to show the uniqueness of the solution.

Keywords: fractional differential equations, integral boundary conditions, gamma function, Banach space, completely continuous operator, existence results.

AMS Subject Classification: 34A08, 34B10.

1. Introduction

The study of fractional differential equations has become a very important and useful area of mathematics over the last few decades due to its numerous applications in various areas of physics, chemistry and engineering such as viscoelasticity [8, 29, 30], dynamical processes in self-similar structures [19], biosciences [20], signal processing [26], systems control theory [33], electrochemistry [25] and diffusion processes [12, 21]. Further, fractional calculus has found many applications in classical mechanics [28] and the calculus of variations [1] and is a very useful and simple means for obtaining solutions to non-homogenous linear ordinary and partial differential equations. For more details, we refer the reader to [23, 24]. There are several approaches to fractional derivatives such as Riemann-Liouville, Caputo,Weyl, Hadamard and Grunwald-Letnikov, etc. Applied problems require those definitions of a fractional derivative that allow the utilization of physically interpretable initial and boundary conditions. The Caputo fractional derivative satisfies these demands, while the Riemann-Liouville derivative is not suitable for mixed boundary conditions. Recently, the theory on existence and uniqueness of solutions of linear and nonlinear fractional differential equations has attracted the attention of many authors, see for example, [2]-[7], [11, 13, 17, 18, 31, 32, 34], [35]-[38] and references therein. However, many of the physical systems can better be described by integral boundary conditions. Integral boundary conditions are encountered in various applications such as population dynamics, blood flow models, chemical engineering and cellular systems. Moreover, boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two-point, three-point, multi-point and nonlocal boundary value problems as special cases, see [3, 10, 14, 15] and references therein.

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Manuscript received December 2012.

In this paper, we study existence and uniqueness of nonlinear fractional differential equations of the type

$$\mathcal{D}^{\alpha}x(t) = f\left(t,\, x(t),\, \mathcal{D}^{\gamma}x(t)\right), \quad \text{for each } t \in J = [0,\ a], \tag{1}$$

satisfying the boundary conditions

$$x(0) + \mu \int_0^a x(s)ds = x(a),\ x'(0) = 0, \tag{2}$$

where $1 \le \alpha < 2,\ 0 < \gamma < \alpha$ and $\mathcal{D}^{\alpha}, \mathcal{D}^{\gamma}$ are the Caputo fractional derivatives.

2. Preliminaries

In this section, we recall some basic definitions and lemmas from fractional calculus [16, 22, 27]. Riemanns modified form of Liouvilles fractional integral operator is a generalization of Cauchys iterated integral formula

$$\int_a^t dt_1 \int_a^{t_1} dt_2 \cdots \int_a^{t_{n-1}} g(t_n)dt_n = \frac{1}{\Gamma(n)} \int_a^t (t-s)^{n-1} g(s)ds \tag{3}$$

where Γ is Eulers gamma function. Clearly, the right-hand side of Eqn. (3) is meaningful for any positive real value of n. Hence, it is natural to define the fractional integral as follows:

Definition 2.1. *If $x \in C([a, b])$ and $\alpha > 0$, then the Riemann-Liouville fractional integral is defined by*

$$\mathcal{I}^{\alpha}_{a^+}x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s)ds. \tag{4}$$

Definition 2.2. *Let $\alpha \in \mathbb{R}, n-1 < \alpha \le n, n \in \mathbb{N}$ and $x \in C((a,\ b),\ \mathbb{R})$, then the Caputo fractional derivative of order α defined by*

$$\mathcal{D}^{\alpha}_{a^+}x(t) = \mathcal{I}^{n-\alpha}_{a^+}\left(\frac{d^n x(t)}{dt^n}\right).$$

We denote $\mathcal{D}^{\alpha}_{a^+}x(t)$ by $\mathcal{D}^{\alpha}_{a}x(t)$ and $\mathcal{I}^{\alpha}_{a^+}x(t)$ by $\mathcal{I}^{\alpha}_{a}x(t)$. Also $\mathcal{D}^{\alpha}x(t)$ and $\mathcal{I}^{\alpha}x(t)$ refer to $\mathcal{D}^{\alpha}_{0^+}x(t)$ and $\mathcal{I}^{\alpha}_{0^+}x(t)$ respectively.
The fractional integral of $x(t) = (t-a)^{\gamma},\ a \ge 0,\ \gamma > -1$ is given as

$$\mathcal{I}^{\alpha}x(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}(t-a)^{\gamma+\alpha}. \tag{5}$$

The following properties of fractional integrals and fractional differential operators will be useful for our further discussion.

$$\mathcal{I}^{\alpha}\left(\mathcal{I}^{\gamma}x(t)\right) = \mathcal{I}^{\gamma}\left(\mathcal{I}^{\alpha}x(t)\right) = \mathcal{I}^{\alpha+\gamma}x(t),\ \ \alpha,\, \gamma \ge 0.$$

The integer order derivative operator $\mathcal{D}^m$ commutes with fractional order $\mathcal{D}^{\alpha}$, i.e,

$$\mathcal{D}^m\left(\mathcal{D}^{\alpha}x(t)\right) = \mathcal{D}^{m+\alpha}x(t) = \mathcal{D}^{\alpha}\left(\mathcal{D}^m x(t)\right). \tag{6}$$

The fractional operator and fractional derivative operator do not commute in general. Then the following result can be found in [9, 16].

Lemma 2.1. [9, 16] *For $\alpha > 0$, the general solution of the fractional differential equation $\mathcal{D}^\alpha x(t) = 0$ is given by*

$$x(t) = \sum_{i=0}^{r-1} c_i t^i, \quad c_i \in \mathbb{R},\ i = 0,\ 1,\ 2,\ \cdots,\ r-1,\ r = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of the real number α.

In view of Lemma 2.1 it follows that

$$\mathcal{I}^\alpha \left(\mathcal{D}^\alpha x(t)\right) = x(t) + \sum_{j=0}^{r-1} c_j t^j \text{ for some } c_j \in \mathbb{R},\ j = 0,\ 1,\ \cdots,\ r-1. \tag{7}$$

But in the opposite way we have,

$$\mathcal{D}^\alpha \left(\mathcal{I}^\gamma(t)\right) = \mathcal{D}^{\alpha-\gamma} x(t). \tag{8}$$

By $C(J,\ \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ and we define $\mathcal{B} = \{x(t) : x(t) \in C[0;1],\ D^\gamma x(t) \in C[0;1]\}$ equipped with the norm

$$\|x(t)\| = \max_{t\in[0,\ 1]} |x(t)| + \max_{t\in[0,\ 1]} |\mathcal{D}^\gamma x(t)|.$$

The space $\mathcal{B}$ is a Banach space [32].

3. Existence and uniqueness of solutions

In the following, we give existence and uniqueness results for fractional differential equation (1) with the integral boundary conditions (2).

Definition 3.1. *A function $x \in \mathcal{B}$ is said to be a solution of equation (1) if x satisfies the equation $\mathcal{D}^\alpha x(t) = f\left(t,\ x(t),\ \mathcal{D}^\gamma x(t)\right)$ on J and the condition (2).*

Lemma 3.1. *Assume that $f : [0,\ a] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. Then $x \in C[0,\ a]$ is a solution of the boundary value problem (1) and (2) if and only if $x(t)$ is the solution of the integral equation*

$$x(t) = \int_0^a \mathcal{G}(t,\ s) f(s,\ x(s),\ \mathcal{D}^\gamma x(s))\, ds, \tag{9}$$

where $\mathcal{G}(t,\ s)$ is the Green's function given by

$$\mathcal{G}(t,\ s) = \begin{cases} \frac{(a-s)^{\alpha-1}}{a\mu\Gamma(\alpha)} - \frac{(a-s)^\alpha - \alpha\, a(t-s)^{\alpha-1}}{a\Gamma(\alpha+1)}, & if\ \ 0 \le s < t, \\ \\ \frac{(a-s)^{\alpha-1}}{a\mu\Gamma(\alpha)} - \frac{(a-s)^\alpha}{a\Gamma(\alpha+1)}, & if\ \ t \le s \le a. \end{cases} \tag{10}$$

Proof. Assume that $x \in C[0,\ a]$ is a solution of the fractional differential equation (1) satisfying boundary conditions (2). Then by Lemma 2.1 and Eqn.(7), we can reduce the problem (1)-(2) to equivalent integral equation

$$\begin{aligned} x(t) &= I^\alpha f(t,\ x(t),\ \mathcal{D}^\gamma x(t)) - c_0 - c_1 t = && (11) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,\ x(s),\ \mathcal{D}^\gamma x(s)) ds - c_0 && (12) \end{aligned}$$

for some constants c_0, c_1. $x'(0) = 0$ yields $c_1 = 0$ and using Fubini's integral theorem, we have

$$\begin{aligned}\int_0^a x(s)ds &= \int_0^a \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u,\, x(u),\, \mathcal{D}^\gamma x(u))du - c_0 \right\} ds = \\ &= \int_0^a \left\{ \frac{1}{\Gamma(\alpha)} \int_0^a (s-u)^{\alpha-1} ds \right\} f(u,\, x(u),\, \mathcal{D}^\gamma x(u))du - c_0 a = \\ &= \int_0^a \frac{1}{\Gamma(\alpha+1)} (a-u)^\alpha f(u,\, x(u),\, \mathcal{D}^\gamma x(u))du - c_0 - c_1 ta.\end{aligned}$$

Applying the boundary condition (2) we obtain $x(0) = -c_0$ and

$$x(a) = \frac{1}{\Gamma(\alpha)} \int_0^a (a-s)^{\alpha-1} f(s,\, x(s),\, \mathcal{D}^\gamma x(s))ds - c_0.$$

Hence

$$c_0 = \frac{1}{a} \int_0^a \frac{(a-s)^\alpha}{\Gamma(\alpha+1)} - \frac{(a-s)^{\alpha-1}}{\mu\Gamma(\alpha)} f(s,\, x(s),\, \mathcal{D}^\gamma x(s))ds.$$

Substituting c_0 into Eqn. (12) we derive

$$\begin{aligned}x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,\, x(s),\, \mathcal{D}^\gamma x(s))ds + \\ &+ \frac{1}{a} \int_0^a \frac{(a-s)^\alpha}{\Gamma(\alpha+1)} - \frac{(a-s)^{\alpha-1}}{\mu\Gamma(\alpha)} f(s,\, x(s),\, \mathcal{D}^\gamma x(s))ds = \\ &= \frac{1}{a} \int_0^t \left\{ \frac{(a-s)^{\alpha-1}}{a\mu\Gamma(\alpha)} - \frac{(a-s)^\alpha - \alpha\, a(t-s)^{\alpha-1}}{a\Gamma(\alpha+1)} \right\} f(s,\, x(s),\, \mathcal{D}^\gamma x(s))ds + \\ &+ \int_t^a \left\{ \frac{(a-s)^{\alpha-1}}{a\mu\Gamma(\alpha)} - \frac{(a-s)^\alpha}{a\Gamma(\alpha+1)} \right\} f(s,\, x(s),\, \mathcal{D}^\gamma x(s))ds = \\ &= \int_0^a \mathcal{G}(t,\ s) f(s,\, x(s),\, \mathcal{D}^\gamma x(s))ds\end{aligned}$$

which completes the proof. □

Theorem 3.1. *Let* $f : [0,\ a] \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous and there exists a function* $\eta : [0,\ a] \to [0,\ \infty]$*, such that* $|f(t,\, x,\, y)| \leq \eta(t)(\lambda_1|x| + \lambda_2|y|)$*,* $\lambda_1,\, \lambda_2 \geq 0$*,* $\lambda_1 + \lambda_2 \leq \delta$*, where* $\delta = \min\{\frac{2\alpha}{m\|\eta\|(3+a)a^\alpha},\, \frac{\alpha\Gamma(\alpha-\gamma+1)}{\|\eta\|a^{\alpha-\gamma}}\}$*,* $m = \max\left\{\frac{1}{a|\mu|\Gamma(\alpha)},\, \frac{1}{a\Gamma(\alpha+1)},\, \frac{1}{\Gamma(\alpha+1)}\right\}$*. Then, the boundary value problem (1)-(2) has a solution.*

Proof. Define an operator $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ by

$$\mathcal{F}x(t) = \int_0^a \mathcal{G}(t,\ s) f(s,\, x(s),\, \mathcal{D}^\gamma x(s))ds. \tag{13}$$

In order to show that the boundary value problem (1)-(2) has a solution, it is sufficient to prove that the operator $\mathcal{F}$ has a fixed point. For $s < t$, equation (10) yields

$$\begin{aligned} |\mathcal{G}(t,\ s)| &\leq \frac{(a-s)^{\alpha-1}}{a|\mu|\Gamma(\alpha)} + \frac{(a-s)^{\alpha}}{a\Gamma(\alpha+1)} + \frac{\alpha\,(t-s)^{\alpha-1}}{\Gamma(\alpha+1)} \leq \\ &\leq m\left\{(a-s)^{\alpha-1} + (a-s)^{\alpha} + (t-s)^{\alpha-1}\right\}. \end{aligned}$$

On the other hand, for $s \geq t$, we have

$$|\mathcal{G}(t,\ s)| \leq m\left\{(a-s)^{\alpha-1} + (a-s)^{\alpha}\right\}.$$

For any $x \in \mathcal{B}$, define the set $\Omega = \{x \in \mathcal{B} : \|x\| \leq \mathcal{R},\ \mathcal{R} > 0\}$. For $x \in \Omega$, under condition on f and using Eqn. (10), we obtain

$$\begin{aligned} |\mathcal{F}x(t)| &= \int\limits_0^a |\mathcal{G}(t,\ s)||f(s,\ x(s),\ \mathcal{D}^{\gamma}x(s))|ds \leq (\lambda_1+\lambda_2)\|x\|\|\eta\| \int\limits_0^a |\mathcal{G}(t,\ s)|ds \leq \\ &\leq (\lambda_1+\lambda_2)\mathcal{R}m\|\eta\| \int\limits_0^a \left\{(a-s)^{\alpha-1} + (a-s)^{\alpha} + (t-s)^{\alpha-1}\right\} ds \leq \\ &\leq \frac{\delta\mathcal{R}m\|\eta\|(a^{\alpha}(1+a) + |t-a|^{\alpha} + t^{\alpha})}{\alpha}. \end{aligned}$$

Hence,

$$\max_{t\in[0,\ a]} |\mathcal{F}x(t)| \leq \frac{\delta\mathcal{R}m\|\eta\|(3+a)a^{\alpha}}{\alpha}.$$

Using Definition 2.2 and Eqn. (10), we have

$$\begin{aligned} |\mathcal{D}^{\gamma}(\mathcal{F}x(t))| &= \left| I^{1-\gamma}\left\{\frac{d\mathcal{F}x(t)}{dt}\right\}\right| = \\ &= \left| I^{1-\gamma}\int\limits_0^a \frac{\partial\mathcal{G}(t,\ s)}{\partial t} f(s,\ x(s),\ \mathcal{D}^{\gamma}x(s))\right| ds \leq \\ &\leq \frac{\alpha-1}{a\Gamma(\alpha+1)} I^{1-\gamma}\int\limits_0^t (t-s)^{\alpha-2}|f(s,\ x(s),\ \mathcal{D}^{\gamma}x(s))|\, ds \leq \\ &\leq \frac{\alpha-1}{a\Gamma(\alpha+1)} I^{1-\gamma}\int\limits_0^t (t-s)^{\alpha-2}\eta(s)\left\{a|x(s)| + b|\mathcal{D}^{\gamma}x(s)|\right\}\, ds \leq \\ &\leq \frac{2(\lambda_1+\lambda_2)\|x\|(\alpha-1)\|\eta\|}{a\Gamma(\alpha+1)} I^{1-\gamma}\int\limits_0^t (t-s)^{\alpha-2}ds \leq \\ &\leq \frac{2(\lambda_1+\lambda_2)\mathcal{R}(\alpha-1)\Gamma(\alpha-1)\|\eta\|}{a\Gamma(\alpha+1)} I^{\alpha-\gamma}(1) = \frac{2\delta\mathcal{R}\|\eta\|}{a\,\alpha\Gamma(\alpha-\gamma+1)} t^{\alpha-\gamma+1}. \end{aligned}$$

Hence,

$$\max_{t\in[0,\ a]} |\mathcal{D}^{\gamma}(\mathcal{F}x(t))| \leq \frac{2\delta\mathcal{R}\|\eta\|a^{\alpha-\gamma}}{\alpha\Gamma(\alpha-\gamma+1)}.$$

Therefore, $\|\mathcal{F}x(t)\| \le \frac{\mathcal{R}}{2} + \frac{\mathcal{R}}{2} = \mathcal{R}$. Thus, $\mathcal{F} : \Omega \to \Omega$. Finally, it remains to show that $\mathcal{F}$ is completely continuous. For any $x \in \Omega$ and for $0 \le t_1 \le t_2 \le a$, we have

$$\begin{aligned}
|\mathcal{F}x(t_2) - \mathcal{F}x(t_1)| &\le \int_0^a |\mathcal{G}(t_2,\ s) - \mathcal{G}(t_1,\ s)||f(s,\ x(s),\ \mathcal{D}^\gamma x(s))|ds \le \\
&\le lm(\lambda_1 + \lambda_2)| \int_0^{t_2} \left((a-s)^{\alpha-1} - (a-s)^\alpha - \alpha\, a(t_2 - s)^{\alpha-1}\right) ds + \\
&+ \int_{t_2}^a \left((a-s)^{\alpha-1} - (a-s)^\alpha\right) ds - \\
&- \int_0^{t_1} \left((a-s)^{\alpha-1} - (a-s)^\alpha - \alpha\, a(t_1 - s)^{\alpha-1}\right) ds - \\
&- \int_{t_1}^a \left((a-s)^{\alpha-1} - (a-s)^\alpha\right) ds| \le lm(\lambda_1 + \lambda_2)|t_1^\alpha - t_2^\alpha|.
\end{aligned}$$

Hence, it follows that $\|\mathcal{F}x(t_2) - \mathcal{F}x(t_1)\| \to 0$, as $t_2 \to t_1$. By Arzela-Ascoli theorem, it follows that $\mathcal{F} : \Omega \to \Omega$ is completely continuous. Thus by Schaurder's fixed point Theorem, the boundary value problem (1)-(2) has a solution. □

Theorem 3.2. *Let $f : [0,\ a] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. If there exists a function $\eta : [0,\ a] \to [0,\ \infty]$, such that such that $|f(t,\ x,\ y) - f(t,\ \tilde{x},\ \tilde{y})| \le \eta(t)(|x - \tilde{x}| + |y - \tilde{y}|)$ for each $t \in [0,\ a]$ and all $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}$ and $2(\alpha - 1)\|\eta\|a^{\alpha-\gamma-1} < \Gamma(\alpha + 1)$. Then the boundary value problem (1)-(2) has a unique solution.*

Proof. Under condition on f, we have

$$\begin{aligned}
|\mathcal{F}x(t) - \mathcal{F}\tilde{x}(t)| &\le \int_0^a |\mathcal{G}(t,\ s)|\,|f(s,\ x(s),\ \mathcal{D}^\gamma x(s)) - f(s,\ \tilde{x}(s),\ \mathcal{D}^\gamma \tilde{x}(s))|\, ds \le \\
&\le m\|\eta\|\|x - \tilde{x}\| \int_0^a \left\{(a-s)^{\alpha-1} + (a-s)^\alpha + (t-s)^{\alpha-1}\right\} ds \le \\
&\le \frac{m\|\eta\|(1-\alpha)|t-a|^\alpha + (1+a)a^\alpha}{\alpha}\|x - \tilde{x}\|.
\end{aligned}$$

Using Eqn. (10) we conclude

$$|\mathcal{D}^\gamma(\mathcal{F}x)(t) - \mathcal{D}^\gamma(\mathcal{F}\tilde{x})(t)| = |I^{1-\gamma}\left\{\frac{d\{(\mathcal{F}x)(t) - (\mathcal{F}\tilde{x})(t)\}}{dt}\right\}| \le$$

$$\le \ I^{1-\gamma} \int_0^a |\frac{\partial \mathcal{G}(t,\ s)}{\partial t}(f(s,\ x(s),\ \mathcal{D}^\gamma x(s)) - f(s,\ \tilde{x}(s),\ \mathcal{D}^\gamma \tilde{x}(s))|\, ds \le$$

$$\le \ \frac{\alpha - 1}{a\Gamma(\alpha+1)} I^{1-\gamma} \int_0^t (t-s)^{\alpha-2}|(f(s,\ x(s),\ \mathcal{D}^\gamma x(s)) - f(s,\ \tilde{x}(s),\ \mathcal{D}^\gamma \tilde{x}(s))|\, ds\, ds \le$$

$$\leq \frac{\alpha-1}{a\Gamma(\alpha+1)} I^{1-\gamma} \int_0^t (t-s)^{\alpha-2}\eta(s)\left\{|x(s)-\tilde{x}(s)| + |\mathcal{D}^\gamma x(s) - \mathcal{D}^\gamma \tilde{x}(s)|\right\} ds \leq$$

$$\leq \frac{2(\alpha-1)\|\eta\| t^{\alpha-\gamma}}{a\Gamma(\alpha+1)} \|x-\tilde{x}\|.$$

Thus, we have

$$\|\mathcal{F}x(t) - \mathcal{F}\tilde{x}(t)\| \leq \frac{2(\alpha-1)\|\eta\| a^{\alpha-\gamma}}{\Gamma(\alpha+1)} \|x-\tilde{x}\|.$$

Therefore, by the contraction mapping Theorem, the boundary value problem (1)-(2) has a unique solution. □

Example 3.1. *Consider the following boundary value problem for nonlinear fractional order differential equation*

$$\mathcal{D}^{\frac{3}{2}} x(t) = \left(3e^t + \upsilon x(t) + \omega \mathcal{D}^{\frac{1}{2}} x(t)\right)^{\frac{1}{3}}, \quad t \in [0,\ 1], \tag{14}$$

$$x'(0) = 0,\ x(0) + \int_0^1 x(t)dt = x(1),$$

where υ, ω *are positive constant and* $\mu = a = 1$, $m = \frac{2}{\sqrt{\pi}}$, $\delta = \frac{3}{4}$. *On other hand* $f(t, x(t), \mathcal{D}^\gamma x(t)) = \sqrt[3]{3e^{-t} + \upsilon x(t) + \omega \mathcal{D}^{\frac{1}{2}} x(t)}$ *satisfies the conditions required in Theorem 3.1, that is* $f(t, x(t), \mathcal{D}^{\frac{1}{2}} x(t)) \leq e^{-t} + \frac{\upsilon}{3}|x(t)| + \frac{\omega}{3}|\mathcal{D}^{\frac{1}{2}} x(t)| \leq (1+e^{-t})\left(\frac{\upsilon}{3}|x(t)| + \frac{\omega}{3}|\mathcal{D}^{\frac{1}{2}} x(t)|\right)$. *Then, Eqn. (14) with assumed boundary conditions has a solution in* Ω *if* $\upsilon + \omega < \frac{9}{4}$.

Example 3.2. *Consider the following boundary value problem for linear fractional order differential equation*

$$\mathcal{D}^{\frac{3}{2}} x(t) = e^{-t^2}\left(x(t) + \mathcal{D}^{\frac{1}{2}} x(t)\right), \quad t \in [0,\ 1], \tag{15}$$

$$x'(0) = 0,\ x(0) + \int_0^1 x(t)dt = x(1).$$

Then, Eqn. (15) with assumed boundary conditions has unique solution in Ω.
In this example $f(t, x(t), \mathcal{D}^\gamma x(t)) = e^{-t^2}\left(x(t) + \mathcal{D}^{\frac{1}{2}} x(t)\right)$ satisfies the conditions required in Theorem 3.2 and moreover $2(\frac{3}{2}-1)\|\eta\| a^{\frac{3}{2}-\frac{1}{2}-1} < \Gamma(\frac{3}{2}+1)$ where $a = 1$, $\eta(t) = e^{-t^2}$.

4. Conclusion

We considered a class of nonlinear fractional order differential equations involving Caputo fractional derivative with lower terminal at 0 in order to study the existence solution satisfying the boundary conditions. The unique solution under Lipschitz condition is also derived. For getting our results in this paper, Schauder's fixed point theorem and Banach contraction principle had played important roles. In order to illustrate our results several examples are presented. Our research work in this paper can be generalized to multiterm nonlinear fractional order differential equations with polynomial coefficients.

5. Acknowledgement

The authors would like to thank to the anonymous referees for their valuable comments and suggestions.

References

[1] Agrawal, O.P., (2002), Formulation of Euler Lagrange equations for fractional variational problems, J. Math. Anal. Appl., 272, pp.368-379.

[2] Agrawal, R.P., Benchora, M., Hamani, S., (2010), A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, Acta Appl. Math., 109 (3), pp.973–1033.

[3] Ahmad, B., Nieto, J., (2009), Existence results for nonlinear boundary value problems of fractional integro differential equations with integral boundary conditions, Boundary Value Problems, 2009, Article ID 708576.

[4] Babakhani, A., Baleanu, D., Khanbabaie, R., (2012), Hopf bifurcation for a class of fractional differential equations with delay, Nonl. Dyn., 69, pp.721-729.

[5] Babakhani, A., Baleanu, D., (2012), Existence and uniqueness of solution for a class of nonlinear fractional order differential equations, Abst. and Appl. Anal., 2012, Article ID 632681.

[6] Babakhani, A., Daftardar-Gejji, V., (2004), Analysis of a system of fractional differential equations, J. Math. Anal. Appl. 293, pp.511-522.

[7] Babakhani, A., Daftardar-Gejji, V., (2003), Existence of positive solutions of nonlinear fractional differential equations, J. Math. Anal. Appl., 278, pp.434-442.

[8] Bagley, R.L., (1983), A theoretical basis for the application of fractional calculus to viscoelasticity, Journal of Rheology, 27 (3), pp.201-210.

[9] Bai, Z.B., Lü, H.S., (2005), Positive solutions of boundary value problems of nonlinear fractional differential equations, J. Math. Anal. Appl. 311, pp.495–505.

[10] Bouncherif, A., (2009), Second order boundary value problems with integral boundary conditions, Nonlinear Analysis, 70 (1), pp.364-371.

[11] Byszewski, L., Lakshmikantham, V., (1991), Theorems about existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a banach space, App. Anal., 40, pp.11–19.

[12] Gorenflo, R., Mainardi, F., (1998), Random walk models for space fractional diffusion processes, Fract. Cal. App. Anal., 12 , 167-191.

[13] Ibrahim, R.W., Momani, S., (2007), On existence and uniqueness of solutions of a class of fractional differential equations, Jou. of Math. Anal. and Appl., 3334, pp.1-10.

[14] Khan, R.A., (2003), The generalized method of quasilinearization and nonlinear boundary value problems with integral boundary conditions, Elec. J. Qualitative Theory of Differential Equations, 10, pp.1-9.

[15] Khan, R.A., (2005), Existence and approximation of solutions of nonlinear problrms with integral boundary conditions, Dynamic Systems and Applications, 14, pp.281-296.

[16] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., (2006), Theory and Applications of Fractional Differential Equations, Elsevier Science, Amsterdam, 204.

[17] Lakshmikantham, V., Vatsala, A.S., (2008), General uniqueness and monotone iterative technique for fractional differential equations, App. Math. Letters, 21, pp.828-834.

[18] Lakshmikantham, V., Vatsala, A.S., (2008), Basic theory of fractional differential equations, Nonlinear Analysis, 26, pp.2677-2682.

[19] Mainardi, F., (1997), Fractals and Fractional Calculus in Continuum Mechanics, Springer, New York.

[20] Magin, R., (2004), Fractional calculus in bioengineering, Crit. Rev. Biom. Eng., 32 (1), pp.1-104.

[21] Metzler, R., Joseph, K., (2000), Boundary value problems for fractional diffusion equations, Physica A, 278, pp.107-125.

[22] Miller K.B., Ross, B., (1993), An Introduction to the Fractional Calculus and Fractional Differential Equations, A Wiely-Interscience Publication, John Wiley & Sons, New York, NY, USA.

[23] Nishimoto, K., (1990), Fractional Calculus and its Applications, Nihon University, Koriyama.

[24] Nishimoto, S.K., Jaw, S., (1993), Applications of fractional calculus to ordinary and partial differential equations of second order, Hiroshima Math. J., 23, pp.63–77.

[25] Oldham, K.B., (2009), Fractional differential equations in electrochemistry, Advances in Engineering Software doi:10.1016/j.advengsoft.2008.12012.

[26] Ortigueira, M., (2003), Special issue on fractional signal processing and applications, Signal Processing, 83 (11), pp.2285-2480.

[27] Podlubny, I., (1999), Fractional Differential Equations, Mathematics in Science and Engineering, 198, Academic Press, New York, London, Toronto.

[28] Rabei, E.M., Alholy, T.S., (2004), Potentials of arbitrary forces with fractional derivatives, Int. J. Mod. Phys A, 19, 1718, pp.3083-3092.

[29] Sorrentinos, G., (2006), Fractional derivative linear models for describing the viscoelastic dynamic behaviour of polymeric beams, Saiont Louis, Messouri, MO proceedings of IMAC.

[30] Sorrentinos, G., (2007), Analytic Modeling and Experimental Identification of Viscoelastic Mechanical Systems, Advances in Fractional Calculus, Springer.

[31] Shuqin, Z., (2006), Existence of Solution for Boundary Value Problem of Fractional Order, Acta Mathematica Scientia, 26B (2), pp.220-228.

[32] Su, X., (2009), Boundary value problem for a coupled system of nonlinear fractional differential equations, Applied Mathematics Letters, 22, pp.64-69.

[33] Vinagre, B.M., Podlubny, I., Hernandez, A., Feliu, V., (2000), Some approximations of fractionalorder operators used in control theory and applications, Fract. Calc. Appl. Anal., 3 (3), pp.231-248.

[34] Wang, C.Y., Wang, R.F., Wang, S., Yang, C.D., (2001), Positive solution of singular boundary value problem for a nonlinear fractional differential equation, Boundary Value Problems, 2011, Article ID 297026.

[35] Wang, C.Y., Yuan, H., Wang, S., (2012), On positive solution of nonlinear fractional differential equation, World Applied Sciences Journal, 18 (11), pp.1540–1545.

[36] Wang, C.Y., Zhang H., Wang, S., (2012), Positive solution of a nonlinear fractional differential equation involving Caputo derivative, Discrete Dynamics in Natural and Society, 2012, Article ID 425408.

[37] Xinwei, S.U., Landong, L., (2007), Existence of solution for boundary value problem of nonlinear fractional differential equation, Appl. Math. j. Chinese Univ. Ser.B, 223 (3), pp.291-298.

[38] Yang, W., (2012), Positive solutions for a coupled system of nonlinear fractional differential equations with integral boundary conditions, Computers and Mathematics with Applications, 63 (1) 1, pp.288–297.

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